Some geometric and functional inequalities related to lower dimensional subspaces

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27 June 2023
INdAM Meeting
"CONVEX GEOMETRY - ANALYTIC ASPECTS"

Overview

1. Overview
2. The Khinchine type inequality
3. The lower bound of $g_{K,m,p}$
4. The properties of $C_{K,m,p}$ and $T_{K,m,p}$
5. Zhang's inequality for lower dimensional subspaces

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- A convex body K is a compact convex subset in the n-dimensional Euclidean space $\mathbb{R}^n$.

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- Denote by $\mathcal{K}^n$ the set of convex bodies in $\mathbb{R}^n$ and $\mathcal{K}_o^n$ the subset of $\mathcal{K}^n$ that convex bodies contain the origin in their interiors.
- If $K \in \mathcal{K}_o^n$, then the polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \quad \text{for all } y \in K\}.$$

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- A star body L in $\mathbb{R}^n$ is a compact star-shaped set about the origin (the intersection of every straight line through the origin with L is a line segment) whose radial function $\rho_L(x) = \max\{\lambda \geq 0 : \lambda x \in L\}$ for $x \in \mathbb{R}^n \setminus \{o\}$ is positive and continuous.

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- Denote the set of star bodies in $\mathbb{R}^n$ by $\mathcal{S}_o^n$.

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- When $m = 1$ and $m = n-1$, $G_{n,1}$ and $G_{n,n-1}$ can be identified as the hemisphere of the unit sphere S^{n-1}.
- Let $\mathrm{P}_\xi : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection map with range space ξ for $\xi \in G_{n,m}$, and $|\cdot|$ denote Lebesgue measure on the corresponding subspace. When not causing confusion, we also write $|x|$ for the Euclidean norm of x in $\xi \in G_{n,m}$.

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- Establishing geometric and functional inequalities related to lower dimensional subspaces is in general not easy.
- The challenge is that the unit sphere S^{n-1} as a hypersurface of $\mathbb{R}^n$ has a globally defined continuous normal vector field, while the Grassmann manifold $G_{n,m}$, $1 < m < n-1$, does not have a similar property.
- "It is not at all clear what is the right body to associate with the function $G_{n,m} \ni \xi \to |\mathrm{P}_\xi K|$, $K \in \mathcal{K}^n$, and in which space it should reside."— E. Milman [JAMS 2023]

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- For example, in order to establish the dual Grassmannian Loomis-Whitney inequality, the authors [L.-Xi-Huang, JLMS 2020] introduced a new functions $g_{K,m,p}$ on Grassmann manifolds, which is a generalization of the Minkowski functional of L_p centroid bodies.

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- For example, in order to establish the dual Grassmannian Loomis-Whitney inequality, the authors [L.-Xi-Huang, JLMS 2020] introduced a new functions $g_{K,m,p}$ on Grassmann manifolds, which is a generalization of the Minkowski functional of L_p centroid bodies.
- The function $g_{K,m,p} : G_{n,m} \to (0, \infty)$ is defined by (up to a factor), for $K \in \mathcal{S}_o^n$ and $\xi \in G_{n,m}$,

$$g_{K,m,p}(\xi) = \left(\frac{1}{|K|} \int_K |\mathrm{P}_\xi z|^p dz \right)^{\frac{1}{p}}, \quad 0 < p < \infty.$$

The L_p-cosine transform

- When $m = 1$, the function $g_{K,m,p}$ reduces to the Minkowski functional of the polar L_p centroid body $\Gamma_p^* K$; i.e., for $u \in S^{n-1}$,

$$\|u\|_{\Gamma_p^* K} = \left(\frac{1}{|K|} \int_K |u \cdot z|^p dz \right)^{\frac{1}{p}}.$$

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- A normalized definition of $\Gamma_p K$ was introduced by Lutwak and Zhang [J. Differential Geom. 1997]. When $p = 1$, $\Gamma_1 K$ is usually written as ΓK, which is the classical centroid body firstly defined and investigated by Blaschke.

The L_p-sine transform

- When $m = n-1$, the function $g_{K,m,p}$ reduces to the Minkowski functional of the polar L_p sine centroid body $\wedge_p^* K$ [Huang-L.-Xi-Ye, JFA 2022]; i.e., for $u \in S^{n-1}$,

$$\|u\|_{\wedge_p^* K} = \left(\frac{1}{|K|}\int_K |\mathrm{P}_{u^\perp} z|^p dz\right)^{\frac{1}{p}}.$$

The L_p-sine transform

- The authors [L.-Xi-Huang, JLMS 2020] showed that the function $g_{K,m,p}$ is continuous on $G_{n,m}$ with respect to the spectral norm. Moreover, an upper bound of $g_{K,m,p}$ for origin-symmetric convex body K in terms of $|K \cap \xi^{\perp}|$ was also obtained, where $K \cap \xi^{\perp}$ is the intersection of K with the orthogonal complement of ξ.

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- One aim of this talk is to continue the study of the properties of $g_{K,m,p}$. Firstly, we shall establish the following Khinchine type inequality (or the inverse Hölder inequality) for m-dimensional subspaces.

Khinchine type inequalities for lower dimensional subspaces

Theorem 1

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function and let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $\Phi(0) = 0$ and be such that Φ and $\Phi(r)/r$ are increasing. Then for $\xi \in G_{n,m}$ and $L \in \mathcal{K}_o^m(\xi)$, we have

$$F(p) := \left(\frac{\int_\xi h(\Phi(\|x\|_L))\|x\|_L^p dx}{\int_\xi h(\|x\|_L)\|x\|_L^p dx} \right)^{\frac{1}{p+m}}$$

is a decreasing function of p on $(-m, +\infty)$ (provided the integrals in $F(p)$ are well defined), that is,

$$F(q) \leq F(p), \qquad q \geq p > -m,$$

with equality if and only if $\Phi(\|x\|_L) = \|x\|_L / F(p)$ for each $x \in \xi$.

Khinchine type inequalities for lower dimensional subspaces

- The case $m = 1$ of Theorem 1 is due to Marshall, Olkin, and Proschan [1967], and a simpler proof was provided by Milman and Pajor [GAFA 1989].

Lemma. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a decreasing function and let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\Phi(0) = 0$ and such that Φ and $\Phi(x)/x$ are increasing. Then*

$$G(p) = \left(\frac{\int_0^\infty h(\Phi(x)) x^p dx}{\int_0^\infty h(x) x^p dx} \right)^{1/(p+1)}$$

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is a decreasing function of p on $]-1, +\infty[$ (provided the integrals in $G(p)$ are well defined).

Khinchine type inequalities for lower dimensional subspaces

Let $h(t) = e^{-t}$. Then we have

Corollary 2

Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a log-concave function ($\log f$ is concave) such that $f(0) = 0$. Then for $\xi \in G_{n,m}$ and $L \in \mathcal{K}_o^m(\xi)$, the function

$$\widetilde{F}(p) := \left(\frac{\int_\xi f(\|x\|_L)\|x\|_L^p dx}{\int_\xi e^{-\|x\|_L}\|x\|_L^p dx} \right)^{\frac{1}{p+m}},$$

is a decreasing function of p on $(-m, +\infty)$ (provided the integrals are well defined). In particular, the function

$$\overline{F}(p) := \left(\frac{1}{m\omega_m \Gamma(p+m)} \int_\xi f(|x|)|x|^p dx \right)^{\frac{1}{p+m}},$$

has the same monotonicity.

Khinchine type inequalities for lower dimensional subspaces

- By the identity

$$\int_K \|\mathrm{P}_\xi z\|_L^p dz = \int_\xi \|x\|_L^p |K \cap (x+\xi^\perp)| dx,$$

we can get another form as follows.

Corollary 3

Let $K \in \mathcal{K}_o^n$, $\xi \in G_{n,m}$ and let $L \in \mathcal{K}_o^m(\xi)$. If $|K \cap \xi^\perp| = \max_{x\in\xi} |K \cap (x+\xi^\perp)|$, then the function

$$\widehat{F}(p) := \left(\frac{\int_K \|\mathrm{P}_\xi z\|_L^p dz}{m|K \cap \xi^\perp||L|B(p+m, n-m+1)} \right)^{\frac{1}{p+m}}$$

is decreasing on $(-m, +\infty)$.

Khinchine type inequalities for lower dimensional subspaces

- The following upper bound of $g_{K,m,p}$ established in [L.-Xi-Huang, JLMS 2020] is a direct consequence of Corollary 3.

Theorem 4, L.-Xi-Huang, JLMS 2020

If K is an origin-symmetric convex body in $\mathbb{R}^n$ and $\xi \in G_{n,m}$, then for $p > 0$

$$g_{K,m,p}(\xi) \leq \frac{|K|^{\frac{1}{m}} B(p+m, n-m+1)^{\frac{1}{p}}}{(m\omega_m |K \cap \xi^{\perp}|)^{\frac{1}{m}} B(m, n-m+1)^{\frac{1}{p}+\frac{1}{m}}}.$$

When $m = 1$, there is equality if and only if K a double cone in the direction ξ.

The lower bound of $g_{K,m,p}$

- The following lemma can be found in [Milman and Pajor, GAFA 1989].

Lemma 5

Let $f : \mathbb{R}^n \to (0, +\infty)$ be a measurable function such that $\|f\|_\infty = 1$ and let $K \in \mathcal{K}_o^n$. Then the function

$$H(p) = \left(\frac{n+p}{n|K|} \int_{\mathbb{R}^n} \|x\|_K^p f(x) dx \right)^{\frac{1}{n+p}}$$

is an increasing function of p on $(-n, +\infty)$. The equality $H(p) = H(q)$ for $p \neq q$ holds if and only if f is the characteristic function of K.

The lower bound of $g_{K,m,p}$

- A direct consequence of Lemma 5 with $n=m$ and $f=\varphi_\xi(x)/\varphi_\xi(0)$, $\varphi_\xi(x)=|K\cap(x+\xi^\perp)|$, is the following theorem.

Theorem 6

Let $p>0$ and $K\in\mathcal{K}_o^n$. For $\xi\in G_{n,m}$, let $L\in\mathcal{K}_o^m(\xi)$. If $|K\cap\xi^\perp|=\max_{x\in\xi}|K\cap(x+\xi^\perp)|$, then

$$\left(\frac{m+p}{m|K|}\int_K\|\mathrm{P}_\xi z\|_L^p dz\right)^{\frac{m}{p}}\geq\frac{|K|}{|L||K\cap\xi^\perp|}, \tag{1}$$

with equality if and only if

$$|K\cap(x+\xi^\perp)|=\begin{cases}|K\cap\xi^\perp|, & \text{if } x\in L;\\ 0, & \text{otherwise.}\end{cases}$$

The lower bound of $g_{K,m,p}$

- When $m = 1$, $L = [-1, 1]$, and K is a symmetric convex body in $\mathbb{R}^n$, inequality (1) reduces to

$$\left(\frac{1+p}{|K|}\int_K |z\cdot\xi|^p dz\right)^{\frac{1}{p}} \geq \frac{|K|}{2|K\cap\xi^\perp|}, \quad \xi \in G_{n,1},$$

with equality if and only if K is a cylinder with height of 2 in the direction of ξ. This has been established by Milman and Pajor [GAFA 1989].

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- Theorems 6 and 4 immediately give the following two-sided inequality. The case $m = 1$ is due to Milman and Pajor.

The lower bound of $g_{K,m,p}$

Theorem 7

If K is an origin-symmetric convex body in $\mathbb{R}^n$ and $\xi \in G_{n,m}$, then for $p > 0$,

$$c_1(m,p)\frac{|K|}{|K\cap\xi^{\perp}|} \leq \left(\frac{1}{|K|}\int_K |\mathrm{P}_{\xi} z|^p dz\right)^{\frac{m}{p}} \leq c_2(m,p)\frac{|K|}{|K\cap\xi^{\perp}|},$$

where

$$c_1(m,p) = \frac{1}{\omega_m}\left(\frac{m}{m+p}\right)^{\frac{m}{p}}$$

and

$$c_2(m,p) = \frac{B(p+m,n-m+1)^{\frac{m}{p}}}{m\omega_m B(m,n-m+1)^{\frac{m}{p}+1}}.$$

When $m = 1$, equality in the left (right)-hand inequality holds if and only if K is a cylinder (double cone).

The lower bound of $g_{K,m,p}$

- In particular, by letting $p \to 0$, we also have

Theorem 8

If K is an origin-symmetric convex body in $\mathbb{R}^n$ and $\xi \in G_{n,m}$, then

$$\frac{e^{-1}}{\omega_m}\frac{|K|}{|K\cap\xi^{\perp}|} \leq \exp\left\{\frac{m}{|K|}\int_K \ln|\mathrm{P}_{\xi}z|dz\right\}$$
$$\leq \frac{|K|}{|K\cap\xi^{\perp}|}\frac{\exp\{(\sum_{k=1}^{m}\frac{m}{k}-1)\}\gamma^{-m}}{m\omega_m B(m,n-m+1)n^m},$$

where γ is the Euler constant.

The lower bound of $g_{K,m,p}$

- As a consequence of Theorem 7 with $p = 2$, we have

Theorem 9

If K is an isotropic origin-symmetric convex body in $\mathbb{R}^n$, then for any $\xi_1, \xi_2 \in G_{n,m}$,

$$\frac{|K \cap \xi_1^{\perp}|}{|K \cap \xi_2^{\perp}|} \leq \binom{n}{m} \left(\frac{(m+1)(m+2)}{(n+1)(n+2)} \right)^{\frac{m}{2}}.$$

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- We say that a star body K in $\mathbb{R}^n$ is isotropic with constant of isotropy L_K if $|K| = 1$ and

$$\int_K |z \cdot u|^2 dz = L_K^2,$$

for every $u \in S^{n-1}$.

The properties of $\widetilde{C}_{K,m,p}$ and $\widetilde{T}_{K,m,p}$

- The following defintions are the radial function of the radial pth mean body T_pK [Gardner and Zhang, Amer. J. Math. 1998] and the p-cross-section body C_pK [Gardner and Giannopoulos, Indiana Univ. Math. J. 1999]. For $u \in S^{n-1}$,

$$\rho_{T_pK}(u) = \Big(\frac{1}{|K|u^\perp|}\int_{K|u^\perp} \big|K \cap (l_u + y)\big|^p dy\Big)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

$$\rho_{C_pK}(u) = \Big(\frac{1}{|K|}\int_{K} \big|K \cap (u^\perp + z)\big|^p dz\Big)^{\frac{1}{p}}, \quad -1 < p < \infty.$$

The properties of $\widetilde{C}_{K,m,p}$ and $\widetilde{T}_{K,m,p}$

- Define the function $\widetilde{T}_{K,m,p} : G_{n,m} \to (0,\infty)$ by, for $K \in \mathcal{K}^n$ and $\xi \in G_{n,m}$,

$$\begin{aligned}\widetilde{T}_{K,m,p}(\xi) &= \Big(\frac{1}{|K|\xi|}\int_{\mathrm{int}K|\xi}\big|K\cap(\xi^\perp+y)\big|^p dy\Big)^{\frac{1}{p}}\\ &= \Big(\frac{1}{|K|\xi|}\int_{\xi}\int_{\xi^\perp}\mathbf{1}_{\mathrm{int}K}(x,y)\big|K\cap(\xi^\perp+y)\big|^{p-1}dxdy\Big)^{\frac{1}{p}}\\ &= \Big(\frac{1}{|K|\xi|}\int_{\mathrm{int}K}\big|K\cap(\xi^\perp+z)\big|^{p-1}dz\Big)^{\frac{1}{p}},\quad 1\le p<\infty,\end{aligned}$$

$$\begin{aligned}\widetilde{T}_{K,m,\infty}(\xi) &= \lim_{p\to\infty}\widetilde{T}_{K,m,p}(\xi) = \max_{y\in\mathrm{int}K|\xi}\big|K\cap(\xi^\perp+y)\big|\\ &= \max_{z\in\mathrm{int}K}\big|K\cap(\xi^\perp+z)\big|,\quad p=\infty.\end{aligned}$$

The properties of $\widetilde{C}_{K,m,p}$ and $\widetilde{T}_{K,m,p}$

- For $K \in \mathcal{K}^n$, denote by $\mathrm{int} K$ the interior of K. Define the function $\widetilde{C}_{K,m,p} : G_{n,m} \to (0, \infty)$ by

$$\widetilde{C}_{K,m,p}(\xi) = \left(\frac{1}{|K|} \int_{\mathrm{int} K} \left| K \cap (\xi^{\perp} + z) \right|^{p} dz \right)^{\frac{1}{p}}, \quad -1 \le p < \infty, p \ne 0,$$

$$\widetilde{C}_{K,m,0}(\xi) = \lim_{p \to 0} \widetilde{C}_{K,m,p}(\xi) = \exp\left(\frac{1}{|K|} \int_{\mathrm{int} K} \log \left| K \cap (\xi^{\perp} + z) \right| dz \right), \quad p$$

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The properties of $\widetilde{C}_{K,m,p}$ and $\widetilde{T}_{K,m,p}$

Theorem 10

The functions $\widetilde{T}_{K,m,p}$ and $\widetilde{C}_{K,m,p}$ are continuous on $G_{n,m}$ with respect to the spectral norm.

Theorem 11

Let $K \in \mathcal{K}^n$ and $\xi \in G_{n,m}$. For $\phi \in \mathrm{GL}(n)$,

$$\widetilde{C}_{\phi K,m,p}(\xi) = |\varepsilon_1| \cdots |\varepsilon_m||\phi| \cdot \widetilde{C}_{K,m,p}(\phi^t \xi),$$

where $\varepsilon_1, \ldots, \varepsilon_m$ is a basis of ξ such that $\phi^t \varepsilon_1, \ldots, \phi^t \varepsilon_m$ is an orthonormal basis of $\phi^t \xi$. In particular, for $O \in \mathrm{O}(n)$,

$$\widetilde{C}_{OK,m,p}(\xi) = \widetilde{C}_{K,m,p}(O^t \xi).$$

The properties of $\widetilde{C}_{K,m,p}$ and $\widetilde{T}_{K,m,p}$

Theorem 12

Let $K \in \mathcal{K}^n$ and $\xi \in G_{n,m}$. For $\phi \in \mathrm{GL}(n)$,

$$\widetilde{T}_{\phi K,m,p}(\xi) = \frac{\left(|\varepsilon_1| \cdots |\varepsilon_m||\phi|\right)^{1-\frac{1}{p}}}{[\phi^{-1}u_1, \ldots, \phi^{-1}u_{n-m}]^{\frac{1}{p}}} \widetilde{T}_{K,m,p}(\phi^t \xi),$$

where $u_1, \ldots, u_{n-m}$ is an orthonormal basis of $\xi^\perp$ and $\varepsilon_1, \ldots, \varepsilon_m$ is a basis of ξ such that $\phi^t\varepsilon_1, \ldots, \phi^t\varepsilon_m$ is an orthonormal basis of $\phi^t\xi$. In particular, for $O \in \mathrm{O}(n)$,

$$\widetilde{T}_{OK,m,p}(\xi) = \widetilde{T}_{K,m,p}(O^t\xi).$$

The monotonicities of $C_{K,m,p}$ and $T_{K,m,p}$

- Define the function $T_{K,m,p} : G_{n,m} \to (0,\infty)$ by

$$T_{K,m,p}(\xi) = \left(\frac{1}{|K|\xi|} \int_{K|\xi} \left| K \cap (\xi^{\perp} + y) \right|^p dy \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$$\begin{aligned} T_{K,m,\infty}(\xi) &= \lim_{p\to\infty} T_{K,m,p}(\xi) = \max_{y \in K|\xi} \left| K \cap (\xi^{\perp} + y) \right| \\ &= \max_{z \in K} \left| K \cap (\xi^{\perp} + z) \right|, \quad p = \infty. \end{aligned}$$

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$$C_{K,m,0}(\xi) = \lim_{p\to 0} C_{K,m,p}(\xi) = \exp\left(\frac{1}{|K|}\int_K \log\left|K\cap(\xi^\perp+z)\right|dz\right), \ p =$$

$$C_{K,m,\infty}(\xi) = \lim_{p\to\infty} C_{K,m,p}(\xi) = \max_{z\in K}\left|K\cap(\xi^\perp+z)\right|, \quad p = \infty.$$

The monotonicities of $C_{K,m,p}$ and $T_{K,m,p}$

- The functions $T_{K,m,p}$ and $C_{K,m,p}$ are both monotonically increasing with respect to p. (The pth mean is increasing with respect to p.)

Theorem 13

Let $K \in \mathcal{K}^n$ and $\xi \in G_{n,m}$. Then

$$\frac{|K|}{|K|\xi|} \le T_{K,m,p}(\xi) \le T_{K,m,q}(\xi) \le \max_{z\in K} \left|K \cap (\xi^{\perp} + z)\right|, \quad 1 \le p \le q. \tag{2}$$

and

$$\frac{|K|}{|K|\xi|} \le C_{K,m,p}(\xi) \le C_{K,m,q}(\xi) \le \max_{z\in K} \left|K \cap (\xi^{\perp} + z)\right|, \quad -1 \le p \le q, \tag{3}$$

Equality holds in each inequality in (2) and (3) if and only if K is the Minkowski sum of an m-dimensional convex body contained in ξ and an $(n-m)$-dimensional convex body contained in $\xi^{\perp}$.

The monotonicities of $C_{K,m,p}$ and $T_{K,m,p}$

A well-known result by Berwald [Acta Math. 1947] (reverse inequalities of the pth mean) implies the following inequalities.

Theorem 14

Let $K \in \mathcal{K}^n$ and $\xi \in G_{n,m}$. Then for $\frac{1}{n-m} \le p \le q$,

$$\max_{z \in K} |K \cap (\xi^\perp + z)| \le \binom{m+qn-qm}{m}^{\frac{1}{q}} T_{K,m,q}(\xi)$$

$$\le \binom{m+pn-pm}{m}^{\frac{1}{p}} T_{K,m,p}(\xi) \le \binom{n}{m} \frac{|K|}{|K|\xi|}.$$

Equality holds in each inequality if and only if $|K \cap (\xi^\perp + y)|^{\frac{1}{n-m}}$ is an affine function of y on $K|\xi$.

The monotonicities of $C_{K,m,p}$ and $T_{K,m,p}$

- To establish the reverse inequalities of $C_{K,m,p}$, the following lemma proved by Borell [Math. Ann. 1973] will be needed.

Lemma 15

Let f be a positive and concave function on a convex body L in $\mathbb{R}^m$. Then the function

$$\psi(p) = \prod_{i=1}^{m}(i+p)\int_L f(x)^p dx$$

is log concave for $p > 0$. Moreover, $\log\psi$ is linear in an interval $[p_0, p_1]$ if and only if f is a roof function over a point in L.

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- For $\tau > 0$ and $x_0 \in K \in \mathcal{K}^n$, the roof function on K with height τ over $x_0 \in K$ is a function $r_{\tau,x_0}(\cdot) : K \to [0, +\infty)$ such that the graph of r_{τ,x_0} in $\mathbb{R}^{n+1}$ is a hyper cone with basis K and height τ, such that the projection of the vertex is $x_0 \in K$.

The monotonicities of $C_{K,m,p}$ and $T_{K,m,p}$

- By Lemma 15, we have

Lemma 16

Let $n > m$ and let f be a positive and concave function on a convex body L in $\mathbb{R}^m$. Then the function

$$\Psi(p) = \left(\frac{\prod_{i=1}^{m}(i + (n-m)(p+1))}{\prod_{i=1}^{m}(i + (n-m))} \frac{\int_L f(x)^{(n-m)(p+1)} dx}{\int_L f(x)^{n-m} dx} \right)^{\frac{1}{p}}$$

is decreasing for $p > -1$, with equality if and only if f is a roof function over a point in L.

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- The case $m = 1$ is due to Gardner and Giannopoulos [Indiana Univ. Math. J. 1999].

The monotonicities of $C_{K,m,p}$ and $T_{K,m,p}$

Theorem 17

Let $K \in \mathcal{K}^n$ and $\xi \in G_{n,m}$. Then for $-1 \le p \le q$,

$$\max_{z \in K} |K \cap (\xi^{\perp} + z)| \le \alpha_{n,m,q} C_{K,m,q}(\xi) \le \alpha_{n,m,p} C_{K,m,p}(\xi) \le \binom{n}{m} \frac{|K|}{|K|\xi|}.$$

Equality holds in each inequality if and only if $|K \cap (\xi^{\perp} + y)|^{\frac{1}{n-m}}$ is an affine function of y on $K|\xi$.

Zhang's inequality for lower dimensional subspaces

- Let $K \in \mathcal{K}^n$, $\xi \in G_{n,m}$, and let $L \in \mathcal{K}_o^m(\xi)$. For $x \in \xi$, define the restricted plane projection function $A_K(\|x\|_L, \xi)$ of K as

$$A_K(\|x\|_L, \xi) = \left|\{\xi^\perp \cap (\xi + y) : |K \cap (\xi + y)|^{\frac{1}{m}} \geq \|x\|_L \text{ for all } y \in \xi^\perp\}\right|$$

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- It is easy to see that $A_K(\|x\|_L, \xi) = 0$ if $\|x\|_L > \sigma(\xi)$ and $\xi^{\perp} \cap (\xi + y)$ is a convex body in $\xi^{\perp}$ if $\|x\|_L < \sigma(\xi)$, where

$$\sigma(\xi) = \max\left\{|K \cap (\xi + y)|^{\frac{1}{m}} : y \in \xi^{\perp}\right\}.$$

Lemma 18

The function $A_K(\|x\|_L, \xi)^{\frac{1}{n-m}}$ is concave with respect to the variable $\|x\|_L$ on $\Omega := \{\|x\|_L : A_K(\|x\|_L, \xi) \neq 0\}, \quad x \in \xi$.

Zhang's inequality for lower dimensional subspaces

- The case $m = 1$ is called the restricted chord projection function introduced by Zhang [Geom. Dedicata 1991].

1. Restricted chord projection of convex bodies

In n-dimensional Euclidean space R^n, let K be a convex body and Σ a hyperplane through the origin. Denote by G any straight line intersecting K. For $\sigma \geqslant 0$, define

$$K'_\Sigma(\sigma) = \{\Sigma \cap G : \Sigma \perp G, |G \cap K| \geqslant \sigma\}.$$

$K'_\Sigma(\sigma)$ is called the restricted chord projection over chord σ of the convex body K on the hyperplane Σ. It can be shown that $K'_\Sigma(\sigma)$ is convex. Obviously, $K'_\Sigma(0)$ is the common orthogonal projection.

The $(n-1)$-dimensional volume of $K'_\Sigma(\sigma)$ in Σ is denoted by $A_K(\sigma, u)$, here u is the unit normal vector of Σ, $A_K(\sigma, u)$ is called the restricted chord projection function of K. It is easy to see that $A_K(\sigma, u) = 0$ if $\sigma > \sigma(u)$ and $K'_\Sigma(\sigma)$ is a convex body in Σ if $\sigma < \sigma(u)$, where $\sigma(u)$ is the maximal chord of K in direction u, i.e.

$$\sigma(u) = \max_G \{\sigma : \sigma = |G \cap K|, G \perp \Sigma\}.$$

Zhang's inequality for lower dimensional subspaces

For $K \in \mathcal{K}^n$, let G be a random m-dimensional plane intersecting K defined by, for $\xi \in G_{n,m}$,

$$G = \{\xi + y : K \cap (\xi + y) \neq \emptyset, \quad y \in \xi^{\perp}\}.$$

Denote by dG the density of G under the group of translations. The integral for the power λ of the planes of K is defined as

$$I_{\lambda,m}(K) = \int_{K \cap G \neq \emptyset} |K \cap G|^{\lambda} dG, \quad \lambda > 0.$$

See the books [Ren, Santaló]. The well-known Crofton-Hadwiger formula says

$$I_{n+1,1}(K) = \frac{n(n+1)}{2} |K|^2.$$

Zhang's inequality for lower dimensional subspaces

Lemma 19

Let $K \in \mathcal{K}^n$, $\xi \in G_{n,m}$, and let $L \in \mathcal{K}^m_o(\xi)$. Then

$$I_{\lambda,m}(K) = \frac{\lambda |G_{n,m}|}{m|L|} \int_{G_{n,m}} \int_{\xi} \|x\|_L^{\lambda-m} A_K(\|x\|_L, \xi) dx d\xi,$$

where

$$|G_{n,m}| = \binom{n}{m} \frac{\omega_n \cdots \omega_{n-m+1}}{\omega_m \cdots \omega_1}.$$

In particular,

$$I_{1,m}(K) = |K||G_{n,m}|.$$

Zhang's inequality for lower dimensional subspaces

By Lemma 18, Lemma 19, and Theorem 1, we have

Theorem 20

Let $K \in \mathcal{K}^n$, $\xi \in G_{n,m}$, and let $L \in \mathcal{K}_o^m(\xi)$. Then, for $p > m$,

$$I_{p,m}(K) \leq c'_{n,m,p}|L|^{-\frac{p}{m}}|G_{n,m}| \int_{G_{n,m}} A_K(0,\xi)^{\frac{m-p}{m}} \left(\int_{\xi} A_K(\|x\|_L, \xi)dx \right)^{\frac{p}{m}} d\xi, \tag{4}$$

where $c'_{n,m,p} = pm^{-\frac{p}{m}} B(p, n-m+1) B(m, n-m+1)^{-\frac{p}{m}}$. Equality in (4) holds if and only if

$$\int_{\xi} A_K(\|x\|_L, \xi)dx = m|L|\sigma(\xi)^m A_K(0,\xi) B(m, n-m+1). \tag{5}$$

Zhang's inequality for lower dimensional subspaces

If $p = n + m$, then we have

Theorem 21

Let $K \in \mathcal{K}^n$, $\xi \in G_{n,m}$, and let $L \in \mathcal{K}_o^m(\xi)$. Then

$$I_{n+m,m}(K) \leq c'_{n,m} |L|^{-\frac{n+m}{m}} |G_{n,m}| \int_{G_{n,m}} A_K(0,\xi)^{-\frac{n}{m}} \left(\int_{\xi} A_K(\|x\|_L, \xi) dx \right)^{\frac{n+m}{m}} d\xi. \tag{6}$$

where

$$c'_{n,m} = (n+m) m^{-\frac{n+m}{m}} B(n+m, n-m+1) B(m, n-m+1)^{-\frac{n+m}{m}}.$$

Equality in (6) holds if and only if (5) holds.

Zhang's inequality for lower dimensional subspaces

When $m = 1$ and $L = B_2^n$, the above theorem immediately recovers Zhang's inequality [Geom. Dedicata 1991].

Zhang's inequality

Let $K \in \mathcal{K}^n$. Then

$$|K||\Pi^* K| \geq \frac{(2n)!}{n^n (n!)^2},$$

with equality if and only if K is a simplex.

Thanks for your attention!